ON THE STABILITY OF SOLUTIONS OF THIRD-ORDER DIFFERENTIAL EQUATIONS

(OB USTOICHIVOSTI RESHENII DIFFERENTSIAL'NYKH URAVNENII TRET'EGO PORIADKA)

PMM Vol.25, No.4, 1961, pp. 785-790

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(Received April 7, 1961)

Liapunov's [1] and Poincaré's [2] qualitative methods in the theory of differential equations lie at the basis of many works on the theory of stability of motion under large initial disturbances.

Erugin has developed a method for the general qualitative investigation of the trajectories in problems on the stability of nonlinear systems. The works [3] and [4] have a direct relation to the present paper. In [5] there is obtained, by means of a qualitative method involving the evaluation of contour integrals, a criterion for the asymptotic stability in the presence of large initial disturbances for second-order nonlinear systems. In the present article such a criterion is found for a third-order nonlinear system. Use is here made of a method which is related to the evaluation of contour integrals.

1. Let us consider the nonlinear system of three equations of the general type

$$\frac{dx_i}{dt} = X_i\,(x_1, x_2, x_3) \qquad (i = 1, 2, 3) \tag{1.1}$$

where X_i is a function possessing continuous second-order partial derivatives in all the variables x_1, x_2 and x_3. (These assumptions relative to the smoothness of the right-hand parts of the system can be weakened.) Furthermore, it is assumed that the origin of the coordinate system is the only state of equilibrium $X_i(0, 0, 0) = 0$.

Let $x_i = x_i(u, t)$, $(i = 1, 2, 3)$, be a one-parameter family of solutions of the system (1.1). We shall use the notation

$$A_i = \sum_{j,k} (-1)^{[i,\,j,\,k]} \frac{\partial x_k}{\partial t}\frac{\partial x_j}{\partial u} \qquad (i = 1, 2, 3)$$

Here the summation is carried out over all possible pairs (j, k) of the numbers 1, 2, 3 that satisfy the conditions: $j \neq k$; $j, k \neq i$. The symbol $[i, j, k]$ will stand for the number of inversions in the permutations of i, j, k

$$B_i = \sum_{j=1}^{3} a_{ij} A_j \qquad (i = 1, 2, 3)$$

Here $\| a_{ij} \|_1^3$ is a constant symmetric matrix possessing positive characteristic numbers

$$\beta_i = B_i / \sqrt{A_1 B_1 + A_2 B_2 + A_3 B_3} \qquad (i = 1, 2, 3)$$

Let us consider some arbitrary surface consisting of integral curves of the system (1.1) and let $x_i = x_i(u, t)$ be its parametric representation. Having taken an arbitrary closed contour Γ on this surface, one can prove with the aid of Green's theorem that the next equation is valid:

$$\oint_\Gamma (\beta_2 X_3 - \beta_3 X_2)\, dx_1 + (\beta_3 X_1 - \beta_1 X_3)\, dx_2 + (\beta_1 X_2 - \beta_2 X_1)\, dx_3 =$$
$$= \iint_\sigma \sum_{i,\,k=1}^{3} \sum_{j=1}^{3} \left(a_{ik} \frac{\partial X_j}{\partial x_j} - a_{jk} \frac{\partial X_i}{\partial x_j} \right) \frac{A_i A_k}{\sqrt{A_1 B_1 + A_2 B_2 + A_3 B_3}}\, du\, dt \qquad (1.2)$$

Here σ is the region of the values (u, t) which corresponds to the part of the surface enclosed by the contour Γ. The double sum appearing in the integrand of Formula (1.2) is a quadratic form in the A_i $(i = 1, 2, 3)$. We shall call it the quadratic form of the system (1.1), corresponding to the matrix $\| a_{ij} \|_1^3$.

We note that one can select for the matrix $\| a_{ij} \|_1^3$ the unit matrix $\| \delta_{ij} \|_1^3$, where $\delta_{ii} = 1$, $\delta_{ij} = 0$ if $i \neq j$. In this case the quadratic form of the system (1.1) corresponding to this matrix will have the form

$$\left(\frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3} \right) A_1^2 + \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_3}{\partial x_3} \right) A_2^2 + \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \right) A_3^2 -$$
$$- \left(\frac{\partial X_1}{\partial x_2} + \frac{\partial X_2}{\partial x_1} \right) A_1 A_2 - \left(\frac{\partial X_1}{\partial x_3} + \frac{\partial X_3}{\partial x_1} \right) A_1 A_3 - \left(\frac{\partial X_2}{\partial x_3} + \frac{\partial X_3}{\partial x_2} \right) A_2 A_3 \qquad (1.3)$$

Theorem. (a) If the solution $x_1 = x_2 = x_3 = 0$ of the system (1.1) is asymptotically stable* relative to disturbances from some neighborhood

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* The verification of this condition can be carried out with the aid of known methods of Liapunov [1].

of the point $x_1 = x_2 = x_3 = 0$;

(b) if one can find a positive symmetric matrix $\| a_{ij} \|_1^3$, possessing positive characteristic numbers, such that the quadratic form of the system (1.1) corresponding to this matrix is nonpositive;

(c) if outside of some sphere, with center at the origin of the coordinates, the right-hand sides of the system (1.1) satisfy the inequality $X_1^2 + X_2^2 + X_3^2 \geqslant q$, where q is a positive constant, then the solution $x_1 = x_2 = x_3 = 0$ of the system (1.1) will be asymptotically stable in the presence of arbitrary initial disturbances.

Proof. By hypothesis (a) of the theorem, the solution $x_1 = x_2 = x_3$ of the system (1.1) is asymptotically stable relative to the disturbances from some (perhaps quite small) neighborhood of the point $x_1 = x_2 = x_3 = 0$. We shall show that under the hypotheses of the theorem the region of attraction of the point $x_1 = x_2 = x_3$ includes the entire space $\{x_1, x_2, x_3\}$. Let us assume the opposite, i.e. that the region of attraction does not include the entire space $\{x_1, x_2, x_3\}$. Then in consequence of the continuous dependence of the solution on the initial conditions, the boundary of the region of attraction is a closed set. Hence, by hypothesis (a), one can find a point (x_{10}, x_{20}, x_{30}) lying on the boundary of the region of stability and nearest to the origin. Let us consider the segment of the radius vector of the point (x_{10}, x_{20}, x_{30}) which contains this point and is of such a small length that the integral curves are not tangent to this segment (such a segment does exist because the integral curves passing through a point intersect its radius vector orthogonally).

Let us further consider the surface consisting of those integral curves that intersect the indicated sigment. With the aid of the one-parameter family of solutions of the system (1.1), the equation of this surface can be written in the form

$$x_i = x_i(u, t) \quad (i = 1, 2, 3)$$

For the sake of definiteness let us assume that the equation of the constructed segment is

$$x_i = x_i(u, 0) \qquad (i = 1, 2, 3) \ u \in [0, 1]$$

Hereby, the smaller values of u correspond to points nearer to the origin of the coordinate system.

We note that the points $x_i = x_i(u, 0)$ $(i = 1, 2, 3)$, $u \in [0, 1]$ of the segment lie in the region of attraction of the point $x_1 = x_2 = x_3 = 0$. In what follows we shall consider that part of the constructed surface which corresponds to positive values of t (for the sake of brevity we shall call it the integral surface).

Through the points of the integral curve $x_i = x_i(1, t)$ $(i = 1, 2, 3)$ we shall construct orthogonal trajectories which lie in the integral surface. Suppose that a certain trajectory, which issues from the point of the integral curve $x_i = x_i(1, t)$ that corresponds to a positive value T of t, intersects some integral curve $x_i = x_i(u_0, t)$, $u_0 \in [0, 1]$ at the point which also corresponds to a positive value of t. Then, starting from an arbitrary point of an integral curve $x_i = x_i(1, t)$ corresponding to some value of $t \geqslant T$, one can construct a segment of an orthogonal trajectory lying on the integral surface. The length of this segment can be made not less than the distance between the positive half-spaces of the integral curves $x_i = x_i(1, t)$ and $x_i = x_i(u_0, t)$. Let this distance be ϵ. Because of the assumption made, $\epsilon > 0$. Let us construct a segment of the orthogonal trajectory, which lies on the integral surface, issues from the point with coordinates $x_i = x_i(1, T)$ $(i = 1, 2, 3)$, and has length ϵ.

Let us mark the integral curve which passes through the end of the constructed segment of the orthogonal trajectory. Suppose that this integral curve is $x_i = x_i(u^*, t)$, $u^* \in [0, 1]$. It is not difficult to see that if one selects in place of T a sufficiently large value, the quantity u^* can be made to lie arbitrarily near 1.

Let us consider a closed contour Γ on the integral surface. Let this contour Γ be formed by the following arcs: segments of the integral curves $x_i = x_i(1, t)$, $x_i = x_i(u^*, t)$; the constructed segment of the orthogonal trajectory of length ϵ; the segment $x_i = x_i(u, 0)$, $u \in [u^*, 1]$.

Let us consider the integral along the contour

$$\oint_{\Gamma} (\beta_2\chi_3 - \beta_3X_2)\ dx_1 + (\beta_3X_1 - \beta_1X_3)\, dx_2 + (\beta_1X_2 - \beta_2X_1)\, dx_3 \tag{1.4}$$

On the basis of Formula (1.2) and condition (b), we conclude that the integral (1.4) is nonpositive. We can also evaluate the integral directly. For this purpose we note that the integrand is the mixed triple product of vectors with the components

$$(X_1,\ X_2,\ X_3)\,(\beta_1,\ \beta_2,\ \beta_3)\,(dx_1,\ dx_2,\ dx_3)$$

Hence, along the segments of the integral curves of the system (1.1) the value of the integral is zero. Along the segment $x_i = x_i(u, 0)$, $u \in [u^*, 1]$, the absolute value of the integral is less than

$$\sqrt{\frac{C_1}{C_2}} \max_{u \in [u^*, 1]} \| X \|_2 \, \| x(1, 0) - x(u^*, 0) \|_2 = N(u^*)$$

Here c_1 is the largest characteristic number of the positive-definite quadratic form $\| B \|_2^{\,2}$ in the A_i $(i = 1, 2, 3)$, while c_2 is the smallest characteristic number of the positive-definite quadratic form

$$S = \sum_{i,\,j=1}^{3} a_{ij}A_iA_j \qquad \| a \|_2 = (a_1^2 + a_2^2 + a_3^3)^{\frac{1}{2}}$$

Along the segment of the orthogonal trajectory the integral is less than $\epsilon\ \sqrt{(qc)_2}$ in consequence of condition (c).

Indeed, rewriting the integral along the segment of the orthogonal trajectory (ϵ) in vector form, we obtain

$$\int_{\varepsilon} (\boldsymbol{\beta} \times \mathbf{X}) \cdot d\mathbf{r} = \int_{\varepsilon} (\mathbf{X} \times d\mathbf{r}) \cdot \boldsymbol{\beta} = \int_{\varepsilon} \| X \|_2 \ \| \beta \|_2 \frac{S}{\| \beta \|_2 \| A \|_2} dr \geqslant \int_{\varepsilon} \sqrt{qc_2}\, dr = \varepsilon \sqrt{qc_2}$$

Combining the obtained estimates, we come to the conclusion that the integral (1.4) satisfies the inequality

$$\oint_{\Gamma} (\boldsymbol{\beta} \times \mathbf{X}) \cdot d\mathbf{r} \geqslant \varepsilon\ \sqrt{qc_2} - N(u^*) \qquad (1.5)$$

Next we note that by an appropriate choice of the contour Γ, the quantity $N(u^*)$ can be made arbitrarily small without changing the value of the curvilinear integral over the corresponding orthogonal trajectory. Along such a contour the integral (1.4) would have a positive value because of the inequality (1.5). This, however, is impossible.

The derived contradiction establishes the theorem.

2. Under the hypotheses of the proved theorem let us evaluate the possible deviations along the trajectories with initial data satisfying the inequality

$$\| x \|_2 \leqslant r_0 \qquad (2.1)$$

Suppose that outside the region (2.1) the quantity $\| X \|_2^{\,2}$ is bounded from below by a positive number q. Because of the continuous dependence of the solutions on the initial data, there exists on the sphere $\| x \|_2 = r_0$ a point (x_{10}, x_{20}, x_{30}) through which there passes an integral curve with a maximum possible deviation

$$\max \| x(t) \|_2 = R \qquad (0 \leqslant t < \infty) \qquad (2.2)$$

The integral curve which passes through the point (x_{10}, x_{20}, x_{30}) intersects the radius vector of this point orthogonally. Indeed, suppose this were not so. We mark the point of the indicated integral curve whose distance from the origin of the coordinates is equal to R. Obviously, the radius vector at this point intersects the integral curve orthogonally. Therefore, due to the fact that the solutions are continuous functions of the initial data, one can find, on the extension of the mentioned radius vector, a point through which passes an integral curve issuing from the region (2.1). This contradicts the definition of the number R.

Let us consider the integral surface formed by the integral curves that intersect a sufficiently small segment of the radius vector of the point (x_{10}, x_{20}, x_{30}); the segment adjoins this point. As above, we write the equation with the aid of a one-parameter family of solutions of the system (1.1) in the form

$$x_i = x_i(u, t), \ (i = 1, 2, 3), \ u \in [0, 1]\cdot$$

Let us establish a property of the orthogonal trajectories on such surfaces. In terms of the Poincaré [2] coordinates u, t, the equation of the orthogonal trajectories on the integral surface can be written in the form

$$x_i = x_i(u, t(u)) \qquad (i = 1, 2, 3)$$

where the function $t(u)$ is a solution of the ordinary differential equation

$$\frac{dt}{du} = -\left(\sum_{i=1}^{3} \frac{\partial x_i}{\partial u} X_i\right) : \sum_{i=1}^{3} X_i^2 \tag{2.3}$$

In the region $0 \leqslant u \leqslant t$, $0 \leqslant t < \infty$, the right-hand side of Equation (2.3) has continuous partial derivatives [6] with respect to t and u. This is a consequence of the smoothness of the right-hand sides of the system (1.1) and of the absence of singular points distinct from the origin. Under these conditions the solution of Equation (2.3) with the initial values u_0, t_0 is either defined for all $u \in [0, 1]$, or it has a vertical asymptote [7] for some value $u \in [0, 1]$. Hence, starting from an arbitrary point of the integral curve $x_i = x_i(1, t)$ $(i = 1, 2, 3)$, one can construct an orthogonal trajectory which lies on the integral surface and either intersects the integral curve $x_i = x_i(0, t)$ $(i = 1, 2, 3)$ or, if extended, will enter the region (2.1). From these considerations it becomes clear that only the first possibility need be considered.

Let us pass an orthogonal trajectory through a point of an integral curve at which Equation (2.2) is satisfied. This trajectory lies on the integral surface and is made to extend to the intersection with the integral curve $x_i = x_i(0, t)$. We denote the distance of the point of intersection from the origin of the coordinates by R_1.

Let us consider the closed contour L formed by the arcs: the constructed segment of the orthogonal trajectory, the segments of the integral curves passing through the ends of this segment, and the radius vector of the point (x_{10}, x_{20}, x_{30}) whose length we shall denote by l_1. A direct appraisal yields

$$\oint_L (\beta_2 X_3 - \beta_3 X_2)\, dx_1 + (\beta_3 X_1 - \beta_1 X_3)\, dx_2 + (\beta_1 X_2 - \beta_2 X_1)\, dx_1 \geqslant \sqrt{qc_2}\,(R - R_1) - N_1$$

$$\left(N_1 = \sqrt{\frac{c_1}{c_2}} \max_{\|x\| \leqslant r_0} \| X \|_2 l_1 \right) \qquad (2.4)$$

Taking into account the fact that the integral on the left of the inequality (2.4) is, because of Formula (1.2) and condition (c), a positive quantity, we obtain the inequality

$$\sqrt{qc_2}\,(R - R_1) - N_1 \leqslant 0 \qquad (2.5)$$

When $R_1 \leqslant r_0$ we find that

$$\sqrt{qc_2}\,(R - r_0) - N_1 \leqslant 0 \qquad (2.6)$$

Bearing in mind that $l_1 < r_0$, we now obtain from (2.6) the estimate

$$R \leqslant r_0 + \frac{N}{\sqrt{qc_2}} \qquad \left(N = \sqrt{\frac{c_1}{c_2}} \max_{\|x\|_2 \leqslant r_0} \| X \|_2 r_0 \right) \qquad (2.7)$$

Suppose that $R_1 > r_0$. Let us consider a region $\| x \|_2 \leqslant r_1$ such that the integral curves with the initial values from this region have the greatest possible deviation, equal to R_1. Let us denote by R_2, l_2 quantities which have analogous meanings with respect to the region $\| x \|_2 \leqslant r_1$ as R_1 and l_1 have with respect to region (2.1). Here again, two cases can arise: $R_2 > r_0$ or $R_2 \leqslant r_0$.

In the first case we have the inequality

$$\sqrt{qc_2}\,(R_1 - R_2) - N_2 \leqslant 0 \qquad \left(N_2 = \sqrt{\frac{c_1}{c_2}} \max_{\|x\|_2 \leqslant r_0} \| X \|_2 l_2 \right) \qquad (2.8)$$

and in the second case the inequality

$$\sqrt{qc_2}\,(R_1 - r_0) - N_2 \leqslant 0 \qquad (2.9)$$

If the inequality (2.9) applies, then we again obtain the estimate (2.7) on the basis of (2.5). Indeed, adding the inequalities (2.5) and (2.9), we obtain

$$\sqrt{qc_2}\,(R - r_0) - (N_1 + N_2) \leqslant 0$$

Noting that $N_1 + N_2 < N$, we can convince ourselves of the validity of the inequality (2.7). Continuing such arguments, we arrive at two possibilities. The first one is this: at some finite step it is found that the corresponding value, say R_n, does not exceed r_0. In this case we have the following inequalities:

$$\sqrt{qc_2}\,(R - R_1) - N_1 \leqslant 0$$
$$\sqrt{qc_2}\,(R_1 - R_2) - N_2 \leqslant 0$$
$$\cdots\cdots\cdots\cdots$$
$$\sqrt{qc_2}\,(R_{n-1} - R_n) - N_n \leqslant 0$$

Adding these inequalities, and bearing in mind that

$$R_n \leqslant r_0,\ N_1 + N_2 + \ldots + N_n < N$$

we again obtain the estimate (2.7). The second possibility consists of the case when $R_n < r_0$ for every n.

We shall show that the first possibility can always be realized. For this purpose it is sufficient to prove that the quantities l_1, l_2, ... can be selected in some bounded closed region not containing the origin, in such a way that they are not less than some positive number.

Let us suppose that the integral curves with initial values from the region $\|x\|_2 \leqslant d$ do not leave the region (2.1). Such a region, obviously, does exist. We consider an arbitrary point of the region $d \leqslant \|x\|_2 \leqslant r_0$. The integral curve which passes through this point has a definite tangent at this point. Hence, it has a perfectly well-defined normal plane P. Because of the continuity of the right-hand sides of the system (1.1), one can find a number δ such that through any point of P which lies within or on the boundary of a sphere with center at the given point and of radius δ there passes an integral curve that intersects P at an angle not less than $1/4\pi$. If δ stands for the largest of the possible values, then δ is a completely determined continuous function in the region $d \leqslant \|x\|_2 \leqslant r_0$. Since the function δ takes on only positive values, its minimum value will also be positive. Let us denote the minimum value of the function δ in the region $d \leqslant \|x\|_2 \leqslant r_0$ by δ_0. It follows from this that the quantities l_1, l_2, ... can be chosen not less than δ_0. For such a choice of the quantities l_1, l_2, ... over a finite number of steps one obtains

$$R_n \leqslant r_0 \quad \left(n \leqslant E\left(\frac{r_0 - d}{\delta_0}\right) + 1\right)$$

This establishes the inequality (2.7).

Note 2.1. An analysis of the proof of the theorem and of the inequality (2.7) will show that for the asymptotic stability of the solution $x_1 = x_2 = x_3 = 0$ of the system (1.1) with respect to initial disturbrances from the region $\|x\|_2 \leqslant r_0$, it is sufficient that condition (b) of the theorem be fulfilled in the region $\|x\|_2 \leqslant R$, where R is any number satisfying the inequality

$$R \geqslant r_0 + \frac{N}{\sqrt{qc_2}}$$

The quantities which appear in this inequality have the same meaning as they had earlier.

Note 2.2. A corollary of a general theorem in [5] (p. 110) gives a criterion for the asymptotic stability of the undisturbed motion with respect to arbitrary initial disturbances. This criterion specifies that the third-order matrix

$$\left\| \frac{\partial X_i}{\partial x_k} + \frac{\partial X_k}{\partial x_i} \right\|_1^3 \tag{2.10}$$

have negative characteristic numbers bounded from above by some negative constant. Computations show that if the matrix (2.10) has this property, then the quadratic form of the system (1.1) which corresponds to the unit matrix (1.3) will be negative-definite. The converse is not true, as can be verified by means of simple examples. Thus, if one selects for the matrix in our theorem the unit matrix $\|\delta_{ij}\|_1^2$, one obtains a criterion of the asymptotic stability for arbitrary initial disturbances, which imposes on the system (1.1) a somewhat weaker condition than the criterion in [5].

3. ***Example*** [8]. Let us consider the system

$$\frac{dx_1}{dt} = x_2 - f(x_1), \qquad \frac{dx_2}{dt} = x_3 - x_1, \qquad \frac{dx_3}{dt} = -ax_1 \tag{3.1}$$

$$a > 0, \qquad f'(x_1) > a \tag{3.2}$$

The system of the first approximation for Equations (3.1) at the point $x_1 = x_2 = x_3 = 0$ has the form

$$\frac{dx_1}{dt} = x_2 - f'(0)\,x_1, \qquad \frac{dx_2}{dt} = x_3 - x_1, \qquad \frac{dx_3}{dt} = -ax_1$$

Because of the condition (3.2), the characteristic equation of this system has roots with negative real parts. Hence, we conclude on the basis of Liapunov's theorem [1, p. 128] that the solution $x_1 = x_2 = x_3 = 0$ of the system (3.1) is asymptotically stable with respect to the initial disturbances from some neighborhood of the point $x_1 = x_2 = x_3 = 0$. The quadratic form of the system (3.1) which corresponds to the matrix

$$\|a_{ij}\|_1^3 = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & a & 2a^2 \end{matrix} \right\|$$

has the form

$$-[f'(x_1)+a]A_2^2-2a^2f'(x_1)A_3^2-2a[f'(x_1)+a]A_2A_3 \qquad (3.3)$$

Making use of condition (3.2), we find on the basis of Sylvester's criterion that the quadratic form of the quantities A_2 and A_3 is negative-definite.

Furthermore, we note that the Jacobian of the system (3.1) is different from zero, equal to $-a$, for all values of x_1, x_2 and x_3. From this it is not difficult to deduce that outside every sphere with center at the origin, the quantity $\|X\|_2$ for the system (3.1) is bounded from below by some positive number. Thus, for the system (3.1), under condition (3.7), there are fulfilled all hypotheses of the proved theorem. Hence, the solution $x_1 = x_2 = x_3 = 0$ of this system is asymptotically stable with respect to arbitrary initial disturbances.

In conclusion, I express my gratitude to N.N. Krasovskii for his attention to and valuable advice on this work.

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Translated by H.P.T.